

ADJUSTING ESTIMATORS FOR MARGINAL DISTRIBUTIONS IN CONTINGENCY TABLES

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ABSTRACT. To take sample biases and skewness in the observations in account, practitioners frequently weight their observations. Considering discrete distributions, the present paper explores the effect of weighted data from an asymptotic point of view. If one marginal is known, weighted estimators for any other marginal distribution are proposed. These have a strictly smaller asymptotic variance whenever there is some dependence between the two marginals. The finite sample performance is illustrated in a simulation study. An application to traffic accident data concludes the paper.

1. INTRODUCTION

Categorical data analysis attracts many fields of research and applications. When applying strategies to real-world data, it seems to be the rule rather than the exception that datasets are lacking to some degree. This is usually due to under-reportings of subgroups of the population, to non-representative study participants, or miscoded observations (see e.g. Lu et al. (2000), Hazell and Shakir (2006), Yamamoto et al. (2008) or Wang et al. (2015), and the applications therein). All these scenarios have in common that they result in a somehow skewed observation distribution. A major task in applied studies is to address the data's skewness at first before proceeding with any other investigation.

The most common way to correct for such a skewness is the application of weighting factors. However, a weighted or cloned sample does not give any additional information on the underlying random mechanisms, a fact that is often ignored by practitioners. Statistical methods have to incorporate the weighting step instead of treating the data clones as new and independent observations. This cloning of dependencies makes the use of weighted samples rather challenging.

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It can be easily seen that weighting or cloning of data can only increase the (asymptotic) variance of distribution estimator compared to an unweighted approach in the univariate case which is briefly discussed in Section 2. In the multi-dimensional case the situation is more interesting. Considering a two-dimensional discrete distribution, let us assume that one marginal distribution is known. This additional information in one category is reasonable in many applications where some properties of the underlying population are detailed reported in large databases and statistical surveys, while data on other categories are collected only in specific (and smaller) studies. Given this additional amount of information, the intriguing question is whether the standard unweighted empirical probability estimators can be improved.

In Section 3 we will settle this question to the positive. Adjusting the observed relative frequencies to the known marginal distribution, we construct an asymptotically unbiased estimator for the unknown marginal. These estimators satisfy a central limit theorem with an asymptotic covariance matrix that is smaller than the asymptotic covariance of the classical estimators (in the sense of positive semi-definite matrices). The weighting is related to the iterative proportional fitting procedure by Deming and Stephan (1940). The information theoretic gain is linked to dependence between the two marginals. It is intuitive that the weighted approach cannot improve the estimator in the case of independent marginals, but whenever there is some dependence structure the proposed method profits in the asymptotic variance.

A simulation study in Section 4 illustrates the finite sample performance of the proposed method. Already for small sample sizes there are substantial improvements for dependent marginals. As a little price to pay, we observe a small finite sample bias.

An exemplary application in accident research is investigated in Section 5. The official national statistics, as e.g. the National Highway Traffic Safety Administration for the United States or the Statistisches Bundesamt for Germany, provide highly accurate information about the accident situation. However, the administrations usually provide only very low dimensional data. To evaluate new driver assistance systems, the automobile producers require further information on more than one accident characteristic. Therefore, they retrieve accident studies of more depth, e.g. the National Automotive Sampling System or the German In-Depth Accident Study (GIDAS). Based on GIDAS data, we estimate the (discretized) distribution of the speed reduction due to collision where the estimators are adjusted for the injury severity distribution from the national statistic.

Section 6 concludes our paper. All technical proofs are deferred to Section 7.

2. ONE-DIMENSIONAL DISTRIBUTIONS

Before we come to our main results in the next Section, we will recall some basic facts for the simple univariate case. We observe an independent and identically distributed (i.i.d.) sample $X_1, \dots, X_n \in \{1, \dots, I\}$ stemming from a discrete probability distribution $\mathbb{P} = (p_1, \dots, p_I)$ with $\sum_{i=1}^I p_i = 1$ and

$$P(X_t = i) = p_i \quad \text{for all } t = 1, \dots, n \text{ and } i = 1, \dots, I.$$

Here and throughout the I possible values, or categories, are labeled by $1, \dots, I$ without loss of generality. The aim is to estimate the probability distribution \mathbb{P} . The usual estimates are computed by the observed relative frequencies

$$\hat{p}_i := \frac{1}{n} \sum_{t=1}^n \mathbb{1}_{\{X_t=i\}} \quad \text{for } i = 1, \dots, I.$$

These estimators satisfy, as $n \rightarrow \infty$, the well-known central limit theorem

$$(2.1) \quad \sqrt{n}((\hat{p}_1, \dots, \hat{p}_I)^\top - (p_1, \dots, p_I)^\top) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma),$$

where $\xrightarrow{\mathcal{D}}$ denotes weak convergence and where the asymptotic covariance matrix $\Sigma = (\Sigma_{rs})_{r,s=1,\dots,I}$ is given by

$$(2.2) \quad \Sigma_{rs} = \begin{cases} -p_r p_s & , r \neq s \\ p_r(1 - p_r) & , r = s \end{cases}.$$

The estimators \hat{p}_i , $i = 1, \dots, I$, can be interpreted as uniformly weighted (or unweighted) means of the observations. Changing the weights from $1/n$ to some general weights $w_t \geq 0$, $t = 1, \dots, n$, with $\sum_{t=1}^n w_t = 1$, yields the more general estimators

$$(2.3) \quad \tilde{p}_i := \sum_{t=1}^n w_t \mathbb{1}_{\{X_t=i\}}, \quad i = 1, \dots, I,$$

for which we observe a slightly different limiting result:

Lemma 2.1. *Let $X_1, \dots, X_n \in \{1, \dots, I\}$ be an i.i.d. sample stemming from a discrete probability distribution $\mathbb{P} = (p_1, \dots, p_I)$, $I \in \mathbb{N}$. Then the estimators $(\tilde{p}_1, \dots, \tilde{p}_I)^\top$ from (2.3) satisfy, as $n \rightarrow \infty$,*

$$(2.4) \quad \frac{1}{\sqrt{\sum_{t=1}^n w_t^2}}((\tilde{p}_1, \dots, \tilde{p}_I)^\top - (p_1, \dots, p_I)^\top) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma),$$

where Σ is given by (2.2).

The proof of the lemma is straightforward and therefore omitted.

By the Cauchy-Schwartz inequality we have $1 = \sum_{i=1}^n w_i \leq \sqrt{\sum_{i=1}^n w_i^2} \sqrt{n}$ and thus $\frac{1}{\sqrt{\sum_{i=1}^n w_i^2}} \leq \sqrt{n}$. Equality holds true if and only if the weights are chosen as $w_t = 1/n$ for all $t = 1, \dots, n$. Consequently, in a univariate i.i.d. framework

non-uniformly weighting of observations always leads to an increased variance and slower convergence rates resulting in wider confidence interval approximations for finite sample sizes. We should emphasize at this point that this property might change when leaving the i.i.d. framework and allow for dependencies between the observations.

Remark 2.2. *In some situations appliers prefer to use data clones instead of (relatively) weighted observations. This may be due to easier implementations when working with databases since it is relatively simple to copy cases. Lemma 2.1 applies immediately to cloned data which can be seen as follows: Let $Z_t \in \mathbb{N}$ for all $t = 1, \dots, n$, be a deterministic sequence with $\sum_{t=1}^n Z_t = N$. Any observation X_t may be cloned Z_t times such that the sample size artificially increases. The new (cloned) sample then reads Y_1, \dots, Y_N , where $Y_{1+\sum_{k=1}^{t-1} Z_k} = \dots = Y_{\sum_{k=1}^t Z_k} = X_t$ for all $t = 1, \dots, n$. The corresponding estimator for p_i is defined by*

$$\check{p}_i = \frac{1}{N} \sum_{r=1}^N \mathbb{1}_{\{Y_r=i\}} = \frac{1}{N} \sum_{t=1}^n Z_t \mathbb{1}_{\{X_t=i\}}.$$

We recover the estimators from (2.3) with weights $w_t := \frac{Z_t}{N}$. Since the minimum variance is obtained for non-cloned observations, i.e. $Z_t = 1$ for all $t = 1, \dots, n$, we conclude that cloning increases variability.

3. CONTINGENCY TABLES

The findings of the previous section directly transfer to the multivariate case. However, we will now investigate whether the situation changes if we have some extra information. Considering a two-dimensional contingency table, we study the estimation of one marginal distribution, say the first, under the assumption that the other marginal distribution, the second, is fully known.

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be a two-dimensional sample taking values in $\{1, \dots, I\} \times \{1, \dots, J\}$ for some $I, J \in \mathbb{N}$ and being distributed according to the law $\mathbb{P} = (p_{ij})_{i=1, \dots, I; j=1, \dots, J}$. The first and second marginals are denoted by $(p_{1\cdot}, \dots, p_{I\cdot})$ and $(p_{\cdot 1}, \dots, p_{\cdot J})$, respectively. Without loss of generality, we can assume $p_{\cdot j} > 0$ for all $j = 1, \dots, J$.

The commonly used estimators for the two-dimensional probabilities p_{ij} are given by

$$\hat{p}_{ij} = \frac{1}{n} \sum_{t=1}^n \mathbb{1}_{\{X_t=i, Y_t=j\}} \quad \text{for all } i = 1, \dots, I, j = 1, \dots, J.$$

The resulting estimator for the first marginal distribution is then defined via

$$\hat{p}_{i\cdot} := \frac{1}{n} \sum_{t=1}^n \mathbb{1}_{\{X_t=i\}} = \sum_{j=1}^J \hat{p}_{ij}, \quad i = 1, \dots, I.$$

In view of (2.1), these estimators satisfy, as $n \rightarrow \infty$,

$$(3.1) \quad \sqrt{n} \left((\hat{p}_{1\cdot}, \dots, \hat{p}_{I\cdot})^\top - (p_{1\cdot}, \dots, p_{I\cdot})^\top \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma)$$

where $\Sigma = (\Sigma_{rs})_{r,s=1,\dots,I}$ is given by

$$(3.2) \quad \Sigma_{rs} = \begin{cases} p_{r\cdot}(1 - p_{r\cdot}) & , r = s \\ -p_{r\cdot}p_{s\cdot} & , r \neq s \end{cases}.$$

The estimators $(\hat{p}_{1\cdot}, \dots, \hat{p}_{I\cdot})^\top$ ignore any information on the second components of the sample. In other words, the estimators treat the two-dimensional sample as a one-dimensional sample. In order to incorporate the additional information to our estimates, we weight the estimators \hat{p}_{ij} such that the known column-wise marginal distribution of the estimators coincide with the true and known marginal distribution $(p_{\cdot 1}, \dots, p_{\cdot J})^\top$. More precisely, we introduce the weighted estimators as

$$(3.3) \quad \tilde{p}_{ij} := \frac{p_{\cdot j}}{\hat{p}_{\cdot j}} \hat{p}_{ij} \quad \text{for any } i = 1, \dots, I; j = 1, \dots, J.$$

Owing to the assumption $p_{\cdot j} > 0$, $j = 1, \dots, J$, we have $\hat{p}_{\cdot j} > 0$ with probability one for a sufficiently large sample size. Hence, due to the above definition we indeed obtain $\tilde{p}_{\cdot j} := \sum_{i=1}^I \tilde{p}_{ij} = p_{\cdot j}$ for all $j = 1, \dots, J$.

The adjusted estimators are fairly simple to implement, but they are not anymore linear in the data owing to the data dependent weights. Since $\hat{p}_{\cdot j}$ is a \sqrt{n} -consistent estimator of $p_{\cdot j}$, all \tilde{p}_{ij} are asymptotically unbiased. The modified contingency table reads

$$\begin{array}{ccc|c} \tilde{p}_{11} & \cdots & \tilde{p}_{1J} & \tilde{p}_{1\cdot} \\ \vdots & & \vdots & \vdots \\ \tilde{p}_{I1} & \cdots & \tilde{p}_{IJ} & \tilde{p}_{I\cdot} \\ \hline p_{\cdot 1} & & p_{\cdot J} & 1 \end{array}$$

where the estimates of interest are given by

$$\tilde{p}_{i\cdot} := \sum_{j=1}^J \tilde{p}_{ij} \quad \text{for all } i = 1, \dots, I.$$

It is worth to mention that the adjusted estimators \tilde{p}_{ij} do not change the cross-product ratios

$$\frac{p_{ij}p_{rs}}{p_{rj}p_{is}}, \quad i, r = 1, \dots, I; j, s = 1, \dots, J, \quad i \neq r, j \neq s,$$

by construction. These ratios describe the degree of association of a contingency table. If, for instance, all cross-product ratios in a contingency table are equal to 1, the table yields independence of the marginals, while values close to zero or infinity describe total dependence. Since the weighted estimators maintain the two-dimensional inner dependence structure for raw data, cross-product ratio-based tests, e.g. for independence (Fisher (1962), Goodman (1964)), can be applied to the modified contingency table without any restrictions and changes in interpretation.

Remark 3.1. *The choice of the weights is in line with a single iteration of the so-called iterative proportional fitting (IPF) procedure by Deming and Stephan (1940). By taking marginal distributions of lower dimension, the IPF algorithm computes a higher dimensional distribution fulfilling the given marginal restrictions (cf. Stephan (1942)). The algorithm works conditional on a specific initialization table. This situation can be interpreted as a two-dimensional sample from which both marginals are known. The procedure aims at adapting the sample characteristics to the given marginals. In the present setup, the weighted sample can be regarded as the result of a single IPF iteration such that the following considerations also hold for any iteration step of the IPF algorithm. For further details on the IPF we refer to Fienberg (1970) and Rüschenendorf (1995).*

Intuitively, the inclusion of further information should at least not worsen the estimators' features. To rigorously evaluate whether there is an improvement from the original estimators $(\hat{p}_i)_{i=1,\dots,I}$ to the modified estimators $(\tilde{p}_i)_{i=1,\dots,I}$, we will compare their limiting variances.

Theorem 3.2. *Let $(X_1, Y_1), \dots, (X_n, Y_n) \in \{1, \dots, I\} \times \{1, \dots, J\}$ be an i.i.d. sample of a two-dimensional distribution $\mathbb{P} = (p_{ij})_{i=1,\dots,I; j=1,\dots,J}$, with some $I, J \in \mathbb{N}$. Then the estimators $\tilde{p}_i = \sum_{j=1}^J \tilde{p}_{ij}$, $i = 1, \dots, I$, with \tilde{p}_{ij} from (3.3) satisfy, as $n \rightarrow \infty$, the limiting result*

$$\sqrt{n}((\tilde{p}_1, \dots, \tilde{p}_I)^\top - (p_1, \dots, p_I)^\top) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Gamma),$$

where $\Gamma = (\Gamma_{k,l})_{k,l=1,\dots,I}$ is given by

$$(3.4) \quad \Gamma_{k,l} = p_k \mathbb{1}_{k=l} - \sum_{j=1}^J \frac{p_{kj} p_{lj}}{p_{\cdot j}}.$$

The quite simple and explicit structure of the asymptotic variance Γ is charming. It can be rigorously understood in terms of the conditional (co-)variance. The latter is defined by

$$\text{Cov}(V, W|Z) := \mathbb{E}[(V - \mathbb{E}[V|Z])(W - \mathbb{E}[W|Z])|Z]$$

for arbitrary random variables V, W and Z on the same probability space. In the following, the notation $\text{Var}(V|Z) := \text{Cov}(V, V|Z)$ also is used. As verified by the

following lemma, the asymptotic covariance matrix Γ of the weighted estimators for the first marginal is given by the covariance of the unweighted estimators conditional on the second component. Hence, we recover in the asymptotic scale that the second margin is known.

Lemma 3.3. *Grant the assumption of Theorem 3.2. We then have for any $k, l \in \{1, \dots, I\}$ and $\Gamma_{k,l}$ from (3.4) that*

$$\frac{1}{n}\Gamma_{k,l} = \mathbb{E}[\text{Cov}(\hat{p}_k, \hat{p}_l | Y_1, \dots, Y_n)].$$

This lemma implies together with the law of total variance that for any $i = 1, \dots, I$

$$\frac{\Sigma_{i,i}}{n} = \text{Var}(\hat{p}_i) = \mathbb{E}[\text{Var}(\hat{p}_i | Y_1, \dots, Y_n)] + \text{Var}(\mathbb{E}[\hat{p}_i | Y_1, \dots, Y_n]) \geq \frac{\Gamma_{i,i}}{n}.$$

Therefore, the asymptotic variance of the adjusted estimators \tilde{p}_i is indeed less or equal to the asymptotic variance of the classical estimators \hat{p}_i . The (asymptotic) information theoretic gain is given by $\text{Var}(\mathbb{E}[\hat{p}_i | Y_1, \dots, Y_n])$. This fact can be generalized to the joint limit law of $(\tilde{p}_1, \dots, \tilde{p}_I)$.

Corollary 3.4. *Grant the assumptions of Theorem 3.2. Then the asymptotic covariance matrices Σ and Γ given in (3.2) and (3.4), respectively, satisfy $\Sigma \geq \Gamma$ in the sense of positive semi-definite matrices: For any vector $c \in \mathbb{R}^I$, we have*

$$(3.5) \quad c^\top (\Sigma - \Gamma) c = \text{Var} \left(\mathbb{E} \left[\sum_{i=1}^I c_i \mathbb{1}_{\{X_1=i\}} \middle| Y_1 \right] \right) \geq 0.$$

In particular, $\Sigma = \Gamma$ holds if and only if (X_1, \dots, X_n) and (Y_1, \dots, Y_n) are independent.

Corollary 3.4 shows that the modification of the estimators by the additional information on the known marginal asymptotically improves the estimation of the target marginal. It should be noted that the theory guarantees improvements for the expected loss, but for a specific sample at hand the weighting approach might be misleading. The degree of improvement depends on the degree of association of the two marginals. To be more precise, the following bound for the overall relative variance reduction can be easily deduced from (3.5):

$$\sum_{i=1}^I \frac{\Sigma_{ii} - \Gamma_{ii}}{\Sigma_{ii}} \geq \sum_{i=1}^I \frac{\Sigma_{ii} - \Gamma_{ii}}{p_{i\cdot}} = \sum_{i=1}^I \sum_{j=1}^J \frac{(p_{ij} - p_{i\cdot} p_{\cdot j})^2}{p_{i\cdot} p_{\cdot j}}.$$

On the right-hand side we recover a natural measure for dependence being the population counterpart of the well known χ^2 -test statistic.

4. NUMERICAL RESULTS

Theorem 3.2 and Corollary 3.4 state that using the modified estimators is asymptotically advantageous over using estimators which ignore the additional information. We have seen that the degree of improvement is related to the degree of association. To study the finite sample effects, we investigate the gain of the modification by simulations in 2x2 contingency tables with fixed marginals by varying the degree of association in terms of the cross-product ratio $cpr = (p_{11}p_{22})/(p_{12}p_{21})$ and several sample sizes n .

We consider three combinations of marginals, namely

- (I) $(p_{1\cdot}, p_{2\cdot}) = (0.5, 0.5)$ and $(p_{\cdot 1}, p_{\cdot 2}) = (0.5, 0.5)$,
- (II) $(p_{1\cdot}, p_{2\cdot}) = (0.9, 0.1)$ and $(p_{\cdot 1}, p_{\cdot 2}) = (0.7, 0.3)$,
- (III) $(p_{1\cdot}, p_{2\cdot}) = (0.2, 0.8)$ and $(p_{\cdot 1}, p_{\cdot 2}) = (0.7, 0.3)$.

We focus on the estimation of $p_{1\cdot}$ in our simulations. To compare the performance of the estimators $\hat{p}_{1\cdot}$ and $\tilde{p}_{1\cdot}$, we simulated for any sample size n and any given cross-product ratio 100,000 samples from which the estimators were computed and noted. Since theory has shown that the variance of $\hat{p}_{1\cdot}$ should be larger than the variance of $\tilde{p}_{1\cdot}$, we denote the average relative proportion of the variance which is removed by the modified estimator $\tilde{p}_{1\cdot}$. The simulation results are given in Figures 1-3. Positive values in the figures indicate whenever the modification of the estimator is -on average- advantageous while negative values indicate that the modification was misleading. Recall that $\log(cpr) = 0$ corresponds to independence of the marginals, while the degree of association increases with the distance from the origin in the horizontal axis.

The simulation results show that for already small and moderate sample sizes the modified estimators lead to performance improvements. It seems that an exception is given in case of independent marginals. In these cases there is a very slight disadvantage for the modified estimator. This effect vanishes when the sample size increases as expected from the theory. Even more important the modified estimators lead to substantial improvements when there is some degree of dependence.

5. CASE STUDY

The assessment of the effectiveness of advanced driver assistance systems plays a crucial role in traffic accident research. For reliable analysis detailed information on the pre-crash phase of an accident has to be known to predict possible benefits of future driver assistant systems. The German In-Depth Accident Study (GIDAS) data contains hundreds of categories which carefully have to be reported for every single accident. Accidents are reported to GIDAS teams by police, if and only if

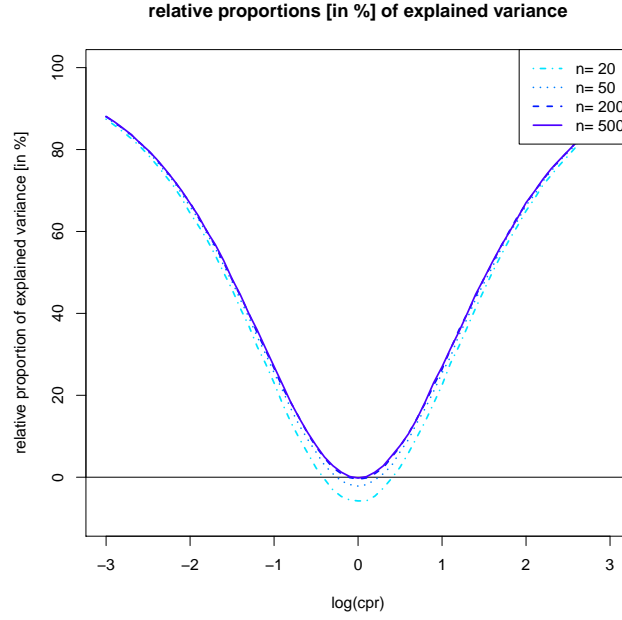


FIGURE 1. Relative proportions (in %) of explained variances conditional on the sample size n and the cross-product ratio cpr for the fixed marginals $(p_{1\cdot}, p_{2\cdot}) = (0.5, 0.5)$ and $(p_{\cdot 1}, p_{\cdot 2}) = (0.5, 0.5)$.

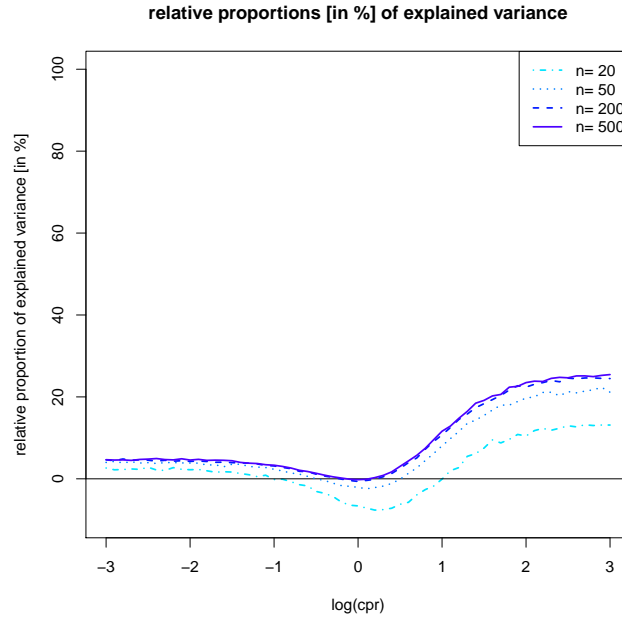


FIGURE 2. Relative proportions (in %) of explained variances conditional on the sample size n and the cross-product ratio cpr for the fixed marginals $(p_{1\cdot}, p_{2\cdot}) = (0.9, 0.1)$ and $(p_{\cdot 1}, p_{\cdot 2}) = (0.7, 0.3)$.

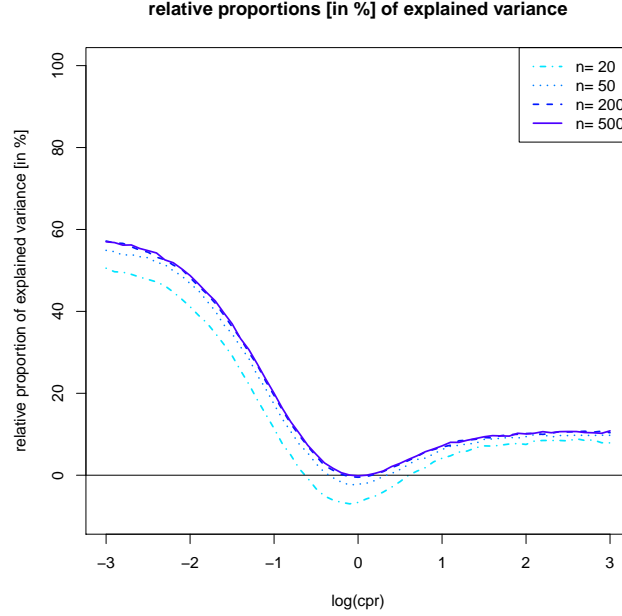


FIGURE 3. Relative proportions (in %) of explained variances conditional on the sample size n and the cross-product ratio cpr for the fixed marginals $(p_{1\cdot}, p_{2\cdot}) = (0.2, 0.8)$ and $(p_{1\cdot}, p_{2\cdot}) = (0.7, 0.3)$.

injured participants are to be expected. This consequently leads to a substantial bias in the injury severity (e.g. Otte et al. (2003), Pfeiffer and Schmidt (2006)). Collision speed or the speed reduction due to the collision, say Δv , is found to be a major correlate to injury severity in traffic accidents.

In this case study we estimate the (discretized) distribution of Δv in passenger car to passenger car collisions with two accident participants. The corresponding GIDAS subsample contains 8,753 cases. To address the skewness in the injury severity of the GIDAS sample, we queried the German National Statistic of the year 2014 (Statistisches Bundesamt (2016)) from which we found the true injury severity distribution for Germany for these accidents (cf. Table 1).

type of injury	slightly injured	severely injured	fatally injured
number of cases	106,181	11,898	423
	[89.6%]	[10.0%]	[0.4%]

TABLE 1. Injury severity distribution in passenger car to passenger car collisions with two accident participants for Germany in 2014 (from Deutsches Bundesamt (2016), p. 100).

	slightly injured	severely injured	fatally injured	total		adjusted	relative difference
$0 \leq \Delta v \leq 10$	346	24	2	372	[11.4%]	12.0%	+5.06%
$11 \leq \Delta v \leq 20$	935	118	1	1054	[32.4%]	33.6%	+3.67%
$21 \leq \Delta v \leq 30$	739	192	4	935	[28.7%]	28.9%	+0.43%
$31 \leq \Delta v \leq 40$	335	154	7	496	[15.2%]	14.7%	−3.49%
$41 \leq \Delta v \leq 50$	124	92	7	223	[6.9%]	6.3%	−7.63%
$51 \leq \Delta v \leq 60$	41	50	6	97	[3.0%]	2.6%	−12.36%
$61 \leq \Delta v \leq 70$	16	25	6	47	[1.4%]	1.2%	−15.75%
$71 \leq \Delta v$	2	21	7	30	[0.9%]	0.7%	−25.89%
total	2,538	676	40	3,254			
	[78.0%]	[20.8%]	[1.2%]		[100%]		

TABLE 2. Two-dimensional data pattern (kind of injury and Δv) from GIDAS in passenger car to passenger car collisions with two accident participants together with marginal estimates for both approaches and their relative difference.

Table 1 can be interpreted as a known marginal distribution for the GIDAS sample. Hence, having the two-dimensional distribution (about the injury severity and Δv) from GIDAS at hand, see Table 2, we can apply the estimation strategy as developed in Section 3. The estimates for the distribution of Δv for the purely GIDAS-based approach, i.e. \hat{p}_i , are contained in the column *total* of Table 2. The National Statistic-aided weighted estimators \tilde{p}_i are given in the column *adjusted* besides its relative difference to the purely GIDAS-based estimates.

From Table 2 we see that the adapted estimates –compared to the ordinary estimates– are increased for lower collision speeds and decreased for higher collision speeds. On a relative scale, the estimate for the highest Δv interval is reduced to around three-quarters of the original estimate. This is not too surprising. It was already stated that GIDAS is biased towards the more severely injured traffic participants, cf. Otte et al. (2003), or compare Table 1 to the marginal distribution in Table 2. Hence, it is to be expected that accidents with higher collision speeds –which are clearly associated with more severe injuries– will be down-weighted. Our proposed method gives precise weights to adjust the more severely injured cases leading to the final outcome in Table 2.

6. CONCLUSIONS

The paper investigated how weighting affects the estimation of a discrete probability distribution. While for one-dimensional data relative weighting will always increase

estimation variances, it has been shown that additional information on a marginal distribution in a contingency table allows for improving the estimation of another marginal if there is some degree of association between the two categories. The gain in terms of the asymptotic (co)variance increases with the degree of dependence. For independent marginals the weighted estimators have the same asymptotic behavior as their classical unweighted counterparts. The simulations indicate a small finite sample bias, but a clear gain when the marginals are substantially associated. Therefore, from theory and from the simulations perspective we suggest to use the adjusted estimators in applications whenever it cannot be assumed that the marginals are independent.

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7. PROOFS

Proof of Theorem 3.2: We first note that the matrix $(\tilde{p}_{ij})_{i=1,\dots,I,j=1,\dots,J}$ can be understood as a function of all \hat{p}_{ij} . Since a limiting result for the joint distribution of all \hat{p}_{ij} is given in (3.1), the asymptotic features of the joint distribution of the weighted estimators \tilde{p}_{ij} can be determined using the delta method.

We define the matrices

$$Z = (\tilde{p}_{ij})_{i=1,\dots,I;j=1,\dots,J} \quad \text{and} \quad W = (\hat{p}_{ij})_{i=1,\dots,I;j=1,\dots,J}$$

and calculate

$$\begin{aligned} Z &= W \cdot \text{diag} \left(\frac{p_{\cdot 1}}{\hat{p}_{\cdot 1}}, \dots, \frac{p_{\cdot J}}{\hat{p}_{\cdot J}} \right) \\ &= W \cdot \text{diag}(\hat{p}_{\cdot 1}, \dots, \hat{p}_{\cdot J})^{-1} \cdot \text{diag}(p_{\cdot 1}, \dots, p_{\cdot J}) \\ &= W \cdot \text{diag}(\mathbf{1}_I^\top \cdot W)^{-1} \cdot \text{diag}(p_{\cdot 1}, \dots, p_{\cdot J}), \end{aligned}$$

where $\mathbf{1}_I \in \mathbb{R}^I$ represents a vector of I repetitions of the value 1. Now interpret

$$\begin{pmatrix} \tilde{p}_{1\cdot} \\ \vdots \\ \tilde{p}_{I\cdot} \end{pmatrix} = Z \mathbf{1}_J = W \cdot \text{diag}(\mathbf{1}_I^\top \cdot W)^{-1} \cdot \begin{pmatrix} p_{\cdot 1} \\ \vdots \\ p_{\cdot J} \end{pmatrix}$$

as a function in W or equivalently $\text{vec}(W) = \text{vec}((\hat{p}_{ij})_{i=1,\dots,I;j=1,\dots,J})$. Hence, $\tilde{P} := (\tilde{p}_{i\cdot})_{i=1,\dots,I}$ can be defined by $\tilde{P} : \mathbb{R}^{IJ} \rightarrow \mathbb{R}^I$, $\text{vec}(W) \mapsto (\tilde{p}_{i\cdot})_{i=1,\dots,I}$. The entries of the associated Jacobi matrix $J_{\tilde{P}} \in \mathbb{R}^{I \times (IJ)}$ are computed in the sequel.

To this end, we denote by \mathbb{I}_{ij} the matrix of zeros with a single 1 at the i -th row and j -th column. We set $u := u_{ij} := i + (j - 1)I$ for any $i = 1, \dots, I$ and $j = 1, \dots, J$

and calculate the u -th column of $J_{\tilde{P}}$ at the point $\text{vec}(W)$ (for brevity we write $\frac{\partial}{\partial W_u}$ instead of $\frac{\partial}{\partial \text{vec}(W)_u}$):

$$\begin{aligned}
& \frac{\partial}{\partial W_u} (\tilde{p}_1, \dots, \tilde{p}_I)^\top \\
&= \left(\frac{\partial}{\partial W_u} W \right) \cdot \text{diag}(\mathbf{1}_I^\top \cdot W)^{-1} \cdot (p_{\cdot 1}, \dots, p_{\cdot J})^\top \\
&\quad + W \cdot \frac{\partial}{\partial W_u} \text{diag}(\mathbf{1}_I^\top \cdot W)^{-1} \cdot (p_{\cdot 1}, \dots, p_{\cdot J})^\top \\
&= \mathbb{I}_{ij} \cdot \text{diag}(\mathbf{1}_I^\top \cdot W)^{-1} \cdot (p_{\cdot 1}, \dots, p_{\cdot J})^\top \\
&\quad + W \cdot \left(\frac{\partial}{\partial W_{i+(j-1)I}} \text{diag} \left(\frac{1}{(\mathbf{1}_I^\top \cdot W)_1}, \dots, \frac{1}{(\mathbf{1}_I^\top \cdot W)_J} \right) \right) \cdot (p_{\cdot 1}, \dots, p_{\cdot J})^\top \\
&= \mathbb{I}_{ij} \cdot \text{diag}(\mathbf{1}_I^\top \cdot W)^{-1} \cdot (p_{\cdot 1}, \dots, p_{\cdot J})^\top \\
&\quad - W \cdot \text{diag} \left(\left(\frac{\mathbb{1}_{r=j}}{(\mathbf{1}_I^\top \cdot W)_r^2} \right)_{r=1, \dots, J} \right) \cdot (p_{\cdot 1}, \dots, p_{\cdot J})^\top \\
&= \mathbb{I}_{ij} \cdot \text{diag}(\mathbf{1}_I^\top \cdot W)^{-1} \cdot (p_{\cdot 1}, \dots, p_{\cdot J})^\top - W \cdot \left(\frac{p_{\cdot 1} \mathbb{1}_{j=1}}{(\mathbf{1}_I^\top \cdot W)_1^2}, \dots, \frac{p_{\cdot J} \mathbb{1}_{j=J}}{(\mathbf{1}_I^\top \cdot W)_J^2} \right)^\top \\
&= \left(\frac{p_{\cdot j} \mathbb{1}_{r=i}}{(\mathbf{1}_I^\top \cdot W)_j} \right)_{r=1, \dots, I} - \frac{p_{\cdot j}}{(\mathbf{1}_I^\top \cdot W)_j^2} (W_{rj})_{r=1, \dots, I} \\
&= \frac{p_{\cdot j}}{\hat{p}_{\cdot j}} (\mathbb{1}_{i=r})_{r=1, \dots, I} - \frac{p_{\cdot j}}{\hat{p}_{\cdot j}^2} (W_{rj})_{r=1, \dots, I}
\end{aligned}$$

For the delta method to apply, we do not need $J_{\tilde{P}}$ at $\text{vec}(W)$, but at its non-estimated target value, i.e. we require $J_{\tilde{P}}$ at $\text{vec}(P) := \text{vec}((p_{ij})_{i=1, \dots, I; j=1, \dots, J})$. By the above computation, the m -th row and $(i + (j - 1)I)$ -th column value is

$$J_{\tilde{P}, m, i+(j-1)I}(\text{vec}(P)) = -\frac{p_{m,j}}{p_{\cdot j}} + \mathbb{1}_{i=m}.$$

Hence, having all derivatives at hand, the limiting covariance matrix reads

$$\Gamma = J_{\tilde{P}} \Sigma J_{\tilde{P}}^\top$$

which we can be calculated component-wise to

$$\begin{aligned}
\Gamma_{m,n} &= \sum_{r=1}^I \sum_{s=1}^J \sum_{t=1}^I \sum_{v=1}^J J_{\tilde{P}, m, r+(s-1)I} \Sigma_{r+(s-1)I, t+(v-1)I} J_{\tilde{P}, n, t+(v-1)I} \\
&= \sum_{r=1}^I \sum_{s=1}^J \sum_{t=1}^I \sum_{v=1}^J \left(-\frac{p_{m,s}}{p_{\cdot s}} + \mathbb{1}_{r=m} \right) (-p_{r,s} p_{t,v} + p_{r,s} \mathbb{1}_{r+(s-1)I=t+(v-1)I}) \\
&\quad \cdot \left(-\frac{p_{n,v}}{p_{\cdot v}} + \mathbb{1}_{t=n} \right)
\end{aligned}$$

$$\begin{aligned}
&= - \sum_{r=1}^I \sum_{s=1}^J \sum_{t=1}^I \sum_{v=1}^J p_{r,s} p_{t,v} \left(-\frac{p_{m,s}}{p_{\cdot s}} + \mathbb{1}_{r=m} \right) \left(-\frac{p_{n,v}}{p_{\cdot v}} + \mathbb{1}_{t=n} \right) \\
&\quad + \sum_{r=1}^I \sum_{s=1}^J p_{r,s} \left(-\frac{p_{m,s}}{p_{\cdot s}} + \mathbb{1}_{r=m} \right) \left(-\frac{p_{n,s}}{p_{\cdot s}} + \mathbb{1}_{r=n} \right) \\
&= - \left(\sum_{r=1}^I \sum_{s=1}^J p_{r,s} \left(-\frac{p_{m,s}}{p_{\cdot s}} + \mathbb{1}_{r=m} \right) \right) \left(\sum_{t=1}^I \sum_{v=1}^J p_{t,v} \left(-\frac{p_{n,v}}{p_{\cdot v}} + \mathbb{1}_{t=n} \right) \right) \\
&\quad + \sum_{r=1}^I \sum_{s=1}^J \frac{p_{rs} p_{ms} p_{ns}}{p_{\cdot s}^2} - 2 \sum_{s=1}^J \frac{p_{ms} p_{ns}}{p_{\cdot s}} + \sum_{s=1}^J p_{ns} \mathbb{1}_{m=n} \\
&= - \left(- \sum_{s=1}^J p_{ms} + p_{m\cdot} \right) \left(- \sum_{r=1}^J p_{nr} + p_{n\cdot} \right) \\
&\quad + \sum_{r=1}^I \sum_{s=1}^J \frac{p_{rs} p_{ms} p_{ns}}{p_{\cdot s}^2} - 2 \sum_{s=1}^J \frac{p_{ms} p_{ns}}{p_{\cdot s}} + \sum_{s=1}^J p_{ns} \mathbb{1}_{m=n} \\
&= \sum_{s=1}^J \frac{p_{ms} p_{ns}}{p_{\cdot s}} - 2 \sum_{s=1}^J \frac{p_{ms} p_{ns}}{p_{\cdot s}} + p_{n\cdot} \mathbb{1}_{m=n} \\
&= p_{n\cdot} \mathbb{1}_{m=n} - \sum_{s=1}^J \frac{p_{ms} p_{ns}}{p_{\cdot s}}.
\end{aligned}$$

□

Proof of Lemma 3.3. Since $(X_t, Y_t)_{t=1, \dots, n}$ is an i.i.d. sample, it suffices to consider $n = 1$. We have for any $k, l = 1, \dots, I$

$$\begin{aligned}
\text{Cov}(\hat{p}_{k\cdot}, \hat{p}_{l\cdot} | Y_1) &= \mathbb{E}[\hat{p}_{k\cdot} \hat{p}_{l\cdot} | Y_1] - \mathbb{E}[\hat{p}_{k\cdot} | Y_1] \mathbb{E}[\hat{p}_{l\cdot} | Y_1] \\
&= \sum_{s=1}^J \sum_{r=1}^J \left(\mathbb{E}[\hat{p}_{ks} \hat{p}_{lr} | Y_1] - \mathbb{E}[\hat{p}_{ks} | Y_1] \mathbb{E}[\hat{p}_{lr} | Y_1] \right).
\end{aligned}$$

Owing to $\hat{p}_{ks} = \mathbb{1}_{\{X_1=k\}} \mathbb{1}_{\{Y_1=s\}}$, we obtain

$$\begin{aligned}
\text{Cov}(\hat{p}_{k\cdot}, \hat{p}_{l\cdot} | Y_1) &= \sum_{s=1}^J \mathbb{1}_{\{Y_1=s\}} \left(\mathbb{E}[\mathbb{1}_{\{X_1=k\}} \mathbb{1}_{\{X_1=l\}} | Y_1] - \mathbb{E}[\mathbb{1}_{\{X_1=k\}} | Y_1] \mathbb{E}[\mathbb{1}_{\{X_1=l\}} | Y_1] \right) \\
&= \sum_{s=1}^J \mathbb{1}_{\{Y_1=s\}} \left(\frac{p_{k,s}}{p_{\cdot s}} \mathbb{1}_{k=l} - \frac{p_{k,s} p_{l,s}}{p_{\cdot s}^2} \right).
\end{aligned}$$

Therefore,

$$\mathbb{E}[\text{Cov}(\hat{p}_{k\cdot}, \hat{p}_{l\cdot} | Y_1)] = p_{k\cdot} \mathbb{1}_{k=l} - \sum_{s=1}^J \frac{p_{k,s} p_{l,s}}{p_{\cdot s}}.$$

□

Proof of Corollary 3.4: In view of (3.2) and (3.4) we have

$$\Sigma - \Gamma = - \left(p_{m\cdot} p_{n\cdot} - \sum_{s=1}^J \frac{p_{ms} p_{ns}}{p_{\cdot s}} \right)_{m,n=1,\dots,I}.$$

We thus obtain for any $c \in \mathbb{R}^I$ that

$$\begin{aligned} c^\top (\Sigma - \Gamma) c &= \sum_{i=1}^I \sum_{j=1}^I c_i c_j \left(\sum_{s=1}^J \frac{p_{is} p_{js}}{p_{\cdot s}} - p_{i\cdot} p_{j\cdot} \right) \\ &= \sum_{s=1}^J \frac{1}{p_{\cdot s}} \left(\sum_{i=1}^I c_i p_{is} \right)^2 - \left(\sum_{i=1}^I c_i p_{i\cdot} \right)^2 \\ &= \sum_{s=1}^J p_{\cdot s} \left(\sum_{i=1}^I c_i \frac{p_{is}}{p_{\cdot s}} \right)^2 - \left(\sum_{i=1}^I c_i p_{i\cdot} \right)^2. \end{aligned}$$

In terms of the function $f_c: \{1, \dots, I\} \rightarrow \{c_1, \dots, c_I\}, i \mapsto c_i$ the last line can be written as

$$\begin{aligned} c^\top (\Sigma - \Gamma) c &= \mathbb{E}[\mathbb{E}[f_c(X_1)|Y_1]^2] - \mathbb{E}[f_c(X_1)]^2 \\ &= \mathbb{E}[\mathbb{E}[f_c(X_1)|Y_1]^2] - \mathbb{E}[\mathbb{E}[f_c(X_1)|Y_1]]^2 \\ (7.1) \quad &= \text{Var}(\mathbb{E}[f_c(X_1)|Y_1]) \\ &\geq 0, \end{aligned}$$

where equality holds if and only if the conditional expectation $\mathbb{E}[f_c(X_1)|Y_1]$ is constant for any function such function f_c .

It remains to verify the equivalence of $\Sigma = \Gamma$ and the independence of X_1 and Y_1 . Let X_1 and Y_1 be independent. Then $\mathbb{E}[\mathbb{E}[f(X_1)|Y_1]] = \mathbb{E}[f(X_1)]$ almost surely for any finite function $f: \{1, \dots, I\} \rightarrow \mathbb{R}$. Therefore, $c^\top (\Sigma - \Gamma) c = 0$ for any $c \in \mathbb{R}^I$, due to (7.1), and thus $\Sigma - \Gamma = 0$. For the other direction suppose (7.1) is zero for any $c \in \mathbb{R}^I$ and let $i \in \{1, \dots, I\}$ and $j \in \{1, \dots, J\}$ be arbitrary. Choosing $c \in \mathbb{R}^I$ as the vector with 1 in the i th row and all other entries zero, we obtain $\mathbb{E}[\mathbb{1}_{\{X_1=i\}}|Y_1] = P(X_1 = i)$ almost surely. We conclude

$$P(X_1 = i, Y_1 = j) = \mathbb{E}[\mathbb{1}_{\{Y_1=j\}} \mathbb{E}[\mathbb{1}_{\{X_1=i\}}|Y_1]] = P(X_1 = i)P(Y_1 = j)$$

for all $i = 1, \dots, I$ and $j = 1, \dots, J$, i.e., X_1 and Y_1 are independent. \square

REFERENCES

- Deming, W.E. and Stephan, F.F. (1940). On a least squares adjustment of a sampled frequency table when the expected marginal totals are known. *The Annals of Mathematical Statistics*, Vol. 11, No. 4, pp. 427-444.
- Fienberg, S.E. (1970): *An iterative procedure for estimation in contingency tables*. The Annals of Mathematical Statistics, Vol. 41, No. 3, pp. 907-917.

- Fisher, R.A. (1962): *Confidence limits for a cross-product ratio*. Australian Journal of Statistics, Vol. 4, p. 41.
- Goodman, L.A. (1964): *Simultaneous confidence limits for cross-product ratios in contingency tables*. Journal of the Royal Statistical Society, Series B, Vol. 26, No. 1, pp. 86-102.
- Hazell, L. and Shakir, S.A.W. (2006). Under-Reporting of Adverse Drug Reactions. *Drug Safety*, Vol. 29, No. 5, pp. 285-396.
- Lu, T.-H., Lee, M.-C. and Chou, M.-C. (2000): Accuracy of cause-of-death coding in Taiwan: types of miscoding and effects on mortality statistics *International Journal of Epidemiology*, Vol. 29, No. 2, pp. 336-343.
- Otte, D., Krettek, C., Brunner, H. and Zwipp, H. (2003): Scientific approach and methodology of a new in-depth-investigation study in Germany so called GIDAS. *Proceedings of the 18th International Technical Conference on the Enhanced Safety of Vehicles*, paper no. 0204, Nagoya (Japan).
- Pfeiffer, M. and Schmidt, J. (2006): Statistical and methodological foundations of the GIDAS accident survey system. *Proceedings of the 2nd International Conference on Expert Symposium on Accident Research*, Hanover, Germany, pp. 8187.
- Rüschendorf, L. (1995): *Convergence of the iterative proportional fitting procedure*. The Annals of Mathematical Statistics, Vol. 23, No. 4, pp. 1160-1174.
- Statistisches Bundesamt (2016): *Verkehr: Verkehrsunfälle 2014*. Statistisches Bundesamt Wiesbaden, Fachserie 8, Reihe 7 (in German), 352 pages.
- Stephan, F.F. (1942): *An iterative method of adjusting sample frequency tables when expected marginal totals are known*. The Annals of Mathematical Statistics, Vol. 13, No. 2, pp. 166-178.
- Wang, W., Rothschild, D., Goel, S. and Gelman, A. (2015): Forecasting elections with non-representative polls. *International Journal of Forecasting*, Vol. 31, No. 3, pp. 980-991.
- Yamamoto, T., Hashiji, J. and Shankar, V.N. (2008): Underreporting in traffic accident data, bias in parameters and the structure of injury severity models. *Accident Analysis and Prevention*, Vol. 40, No. 4, pp. 1320-1329.

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